Unit-3 (Sequences and Series)

1) **Sequences** :→ A sequence is a function $f: N \to S$, whose domain is the set N of all natural numbers whereas the range may be any set S.

Real sequence A Real sequence is a function $f: N \to R$, whose domain is Natural numbers N and range a subset of set R.

A sequence is denoted by $\langle x_n \rangle$ or $\{x_n\}$

$$\langle x_n \rangle = \{x_1, x_2, x_3 \dots x_n \dots\}$$

x_1 : first term of seq; x_2 : second term of seq; x_n : nth term of seq.

Range of sequence :→ Set of all distinct elements of sequence is called Range.

Ex. $\langle (-1)^n \rangle = \{-1, 1, -1, 1, -1, 1 \dots\}$, $n \in N$

Range $= \{-1, 1\}$ finite set

Constant sequence A sequence defined $\{x_n\}$ defined as $\{x_n\} = c$, $c \in R$, $\forall n \in N$, where c is const. is called constant sequence.

$\{x_n\} = \{c, c, c, \dots\}$ → Constant sequence.

Range $= \{c\}$ finite set.

(2) Convergent, Divergent and Oscillating sequence :-

Convergent Sequence :→ A sequence $\{a_n\}$ is said to be convergent if $\lim_{n\to\infty} a_n$ is finite

ex. $\langle \frac{1}{2^n} \rangle = \{ \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3} \cdots \frac{1}{2^n} \cdots \}$

$a_n = \frac{1}{2^n}$, $\lim_{n\to\infty} \frac{1}{2^n} = 0$ (finite) $\Rightarrow$ cgt. seq.

Divergent sequence :→ A sequence $\{a_n\}$ is said to be divergent if $\lim_{n\to\infty} \{a_n\}$ is not finite

i.e $\lim_{n\to\infty} a_n = \infty$ or $-\infty$.

ex: $\langle n^2 \rangle = \{1, 2^2, 3^2 \cdots\}$

$\lim_{n\to\infty} n^2 = \infty \Rightarrow$ seq^n is dgt.

Oscillatory sequence :→ If a sequence $\{a_n\}$ neither converges to finite number nor diverges to ∞ or $-\infty$ it is called Oscillatory sequence.

ex. $\langle (-1)^n \rangle = \{-1, 1, -1, 1, -1, 1 \cdots\}$

$\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} (-1)^{2n} = 1$ if n is even

$\lim_{n\to\infty} a_{2n+1} = \lim_{n\to\infty} (-1)^{2n+1} = -1$, if n is odd

Thus $\lim_{n\to\infty} a_n$ does not exist $\Rightarrow$ seq^n does not cgt.

Hence Oscillates finitely.

ex. $\langle (-1)^n n \rangle = \{-1, 2, -3, 4, -5 \ldots\}$

$$\left.\begin{array}{l} \lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} (-1)^{2n} \cdot 2n = \infty \\ \lim_{n\to\infty} a_{2n+1} = \lim_{n\to\infty} (-1)^{2n+1}(2n+1) = -\infty \end{array}\right\} \text{does not cgt.}$$

Hence Oscillates infinitely.

(3) Bounded sequence A sequence is said to be bounded if $\exists$ two real numbers k and K $(k \leq K)$ such that $k \leq a_n \leq K$. $\forall\, n \in N$ OR $|a_n| \leq M$ $\forall\, n \in N$, $M \in R$

ex. $\langle \frac{1}{n} \rangle = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \ldots\}$

$0 < a_n \leq 1 \quad \forall\, n \in N$

$\Rightarrow \langle \frac{1}{n} \rangle$ is bounded sequence.

Unbounded Sequence :→ If $\exists$ no real number M such that $|a_n| \leq M$, $\forall\, n \in N$, then the seq $\langle a_n \rangle$ is said to be unbounded.

ex. $\langle 2^{n-1} \rangle = \{1, 2, 2^2, 2^3 \ldots\}$

Here $a_n \geq 1$, $\forall\, n \in N$, $\exists$ no real number K such that $a_n \leq K$

$\Rightarrow$ Sequence is Unbounded above.

(ii) $\langle (-1)^n n \rangle = \{-1, 2, -3, 4, -5, 6 \ldots\}$

$\exists$ no real number K such that $|a_n| \leq K$ $\Rightarrow$ Seq is Unbounded.

(4) Monotonic sequence :→ A sequence is said to be Monotonic if it is either monotonically increasing or monotonically decreasing sequence. Now

A sequence $\{a_n\}$ is said to be monotonically increasing if $a_{n+1} \geq a_n \;\forall\; n \in N$

A sequence $\{a_n\}$ is said to be monotonically decreasing if $a_{n+1} \leq a_n \;\forall\; n \in N$

Ex. (i) $\langle \frac{n}{n+1} \rangle = \{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4} \cdots \frac{n}{n+1} \cdots \}$

$\frac{1}{2} < \frac{2}{3} < \frac{3}{4} < \cdots$ ⇒ monotonic increasing

Now $\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n}(1+\frac{1}{n})} = 1$ finite ⇒ cgt seq.

(ii) $\langle n \rangle = \{1, 2, 3, 4 \cdots \}$

$1 < 2 < 3 \cdots$ ⇒ monotonic increasing seq.

$\lim_{n\to\infty} n = \infty$ (infinite) ⇒ dgt. seq.

Ex. (i) $\langle \frac{1}{n} \rangle = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \cdots \}$

$1 > \frac{1}{2} > \frac{1}{3} > \frac{1}{4} \cdots$ ⇒ monotonic decreasing seq.

$\lim_{n\to\infty} \frac{1}{n} = 0$ finite ⇒ cgt seq.

(ii) $\langle -n \rangle = \{-1, -2, -3, -4 \cdots \}$

$-1 > -2 > -3 > -4 \cdots$ ⇒ monotonic dereasing seq

Now $\lim_{n\to\infty} (-n) = -\infty$ ⇒ dgt seq.

Q→ Discuss the convergence of the following sequence.

(a) $\langle a_n \rangle = \langle \frac{n+1}{n} \rangle$

Sol^n $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} \frac{\cancel{n}(1+\frac{1}{n})}{\cancel{n}}$

$= 1$ finite.

$\Rightarrow$ seq is convergent

(b) $\langle \frac{n^2+1}{n+1} \rangle$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2+1}{n+1} = \lim_{n\to\infty} \frac{n^2(1+\frac{1}{n^2})}{n(1+\frac{1}{n})}$

$= \lim_{n\to\infty} \frac{n(1+\frac{1}{n^2})}{(1+\frac{1}{n})} = \frac{\infty(1+0)}{1+0} = \infty$

$\Rightarrow$ sequence is divergent

(c) $a_n = 1 + \frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^n}$

$a_n = 1 + \frac{1}{3} + \frac{1}{3^2} + \cdots \frac{1}{3^n}$ (G.P.) ($r = 1/3 < 1$)

$= \frac{1}{3^0} + \frac{1}{3^1} + \frac{1}{3^2} \cdots + \frac{1}{3^n}$

$a_n = \frac{a(1-r^{n+1})}{1-r} = \frac{1(1-\frac{1}{3^{n+1}})}{1-\frac{1}{3}} = \frac{3}{2}\left(1-\frac{1}{3^{n+1}}\right)$

Now $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3}{2}\left(1-\frac{1}{3^{n+1}}\right) = \frac{3}{2}$ finite

$\Rightarrow$ seq is cgt.

Series

$\langle u_n \rangle = \{u_1, u_2, u_3, u_4 \dots u_n \dots\}$ is sequence and $u_1, u_2, u_3 \dots u_n$ is called first, second, third --- n^{th} term of sequence.

An expression of form

$u_1 + u_2 + u_3 \dots + u_n + \dots$ is called series

so $u_1 + u_2 + u_3 \dots + u_n + \dots = \sum_{k=1}^{\infty} u_k \Rightarrow$ Infinite series

$u_1 + u_2 + u_3 + \dots u_n = \sum_{r=1}^{n} u_r \Rightarrow$ finite series.

Now $S_n =$ first n term's sum of series.

$S_n = u_1 + u_2 + u_3 + \dots + u_n = \sum_{n=1}^{n} u_n$

Convergence of a series :→

⇒ If $\lim_{n\to\infty} S_n = S$ (finite and Unique) then $\sum u_n$ is Convergence

⇒ If $\lim_{n\to\infty} S_n = \infty$ or $-\infty$ then $\sum u_n$ is divergent.

⇒ If $\lim_{n\to\infty} S_n$ is neither finite Unique, nor ∞ or $-\infty$, then $\sum u_n$ is called Oscillatory series.

$\sum u_n$ is said to Oscillate finitely if $\lim_{n\to\infty} S_n$ is finite but not Unique and Oscillates b/w finite limits.

$\sum u_n$ is said to Oscillate infinitely if $\lim_{n\to\infty} S_n$ Oscillates between ∞ and $-\infty$

Note:→ An Geometric series with common ratio r is.

(i) convergent if $|r|<1$
(ii) divergent if $r>1$
(iii) Oscillate finitely $r=-1$
(iv) Oscillate infinitely $r<-1$

Q→ Test the convergence of the following infinite series

$$1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+------$$

Sol Here $S_n = 1+\frac{1}{3}+\frac{1}{3^2}+---n$ terms.

$$S_n = \frac{a(1-r^n)}{1-r}, \quad r=\frac{1}{3}<1$$

$$= \frac{1\left(1-\frac{1}{3^n}\right)}{1-\frac{1}{3}} = \frac{3}{2}\left(1-\frac{1}{3^n}\right)$$

Now $\lim_{n\to\infty} S_n = \lim_{n\to\infty}\frac{3}{2}\left(1-\frac{1}{3^n}\right) = \frac{3}{2}$ (finite Unique)

⇒ Series is convergent.

Q→ Test convergence of the following infinite series.

$$2+2^2+2^3+2^4+----$$

Sol Here $S_n = \frac{2(2^n-1)}{2-1}$ $\left[\begin{array}{l} r=2>1. \\ S_n = \frac{a(r^n-1)}{r-1} \end{array}\right]$

$$= 2(2^n-1) = (2^{n+1}-2)$$

Now $\lim_{n\to\infty}(2^{n+1}-2) = \infty$ infinite

⇒ Series divergent.

Note :→

$\sum U_n$ to be convergent, it is necessary but not sufficient that $\boxed{\lim_{n\to\infty} U_n = 0}$

Fundamental Properties :→

⇒ The nature of series remains unaltered if signs of all terms are altogether changed.

⇒ The convergence or divergence of series remains uneffected if a finite number of terms are added or neglected

⇒ If each term of a series is multiplied or divided by some fixed quantity other than zero, then new series so obtained will remain convergent or divergent according as it was orginally convergent or divergent.

Practice Questions

Q-1 Test convergence of following series

(a) $1+2+3+4+5+ - - - - -$ [Ans:- divergent]

(b) $1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + - - - - -$ [Ans:- Convergent]

(c) $1+3+5+7+ - - - - -$ [Ans:- Divergent]

p-series Test OR Hyper-Harmonic Test

$$\sum \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots + \frac{1}{n^p} + \cdots$$

is $\Rightarrow$ convergent if $p > 1$

$\Rightarrow$ divergent if $p \leq 1$

Comparision Test for Positive Term Series :-

If $\sum u_n$ and $\sum v_n$ be two positive term series such that $\lim_{n \to \infty} \left(\frac{u_n}{v_n}\right) = l$ (fixed, finite, non zero) $\Rightarrow$ Test is Applic-able

then both series will converge or diverge simultaneously.

Here $\sum v_n$ is called Auxiliary series whose convergence or divergence is already known.

Method for finding Auxiliary series :-

If we have u_n in the form

$$u_n = \frac{A_1 n^p + A_2 n^{p-1} + \cdots}{B_1 n^q + B_2 n^{q-1} + \cdots}$$

Then $\boxed{v_n = \frac{n^p}{n^q} = \frac{1}{n^{q-p}}}$

Q-1 Test convergence of series.

$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+ - - - -$

Sol $u_n = \frac{1}{\sqrt{n}}$

$\Sigma \frac{1}{\sqrt{n}} = 1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+ - - - -$

Here $p = 1/2 < 1 \Rightarrow$ series is divergent [compare with p-series]

Q → Test convergence of series.

$1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+ - - -$

Sol $\Sigma u_n = 1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+ - - - + \frac{1}{2n-1} + - -$

1, 3, 5, 7 - - - A.P
n^{th} term $= a+(n-1)d$
$= 1+(n-1)2$
$= 2n-1$

$u_n = \frac{1}{2n-1}$

Aux. Series $v_n = \frac{1}{n}$, [n^{th} term of Aux. series Σv_n].

Now $\frac{u_n}{v_n} = \frac{1}{2n-1}\cdot\frac{n}{1} = \frac{n}{2n-1}$

Now $\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty}\frac{n}{2n-1} = \lim_{n\to\infty}\frac{\cancel{n}}{\cancel{n}(2-\frac{1}{n})} = \lim_{n\to\infty}\frac{1}{2-\frac{1}{n}}$

$= \frac{1}{2}$

(finite, non zero)

⇒ Comparision Test is applicable.

Now by p-test. $\Sigma v_n = \Sigma\frac{1}{n}$ is divergent as $p=1$

Hence Σu_n is dgt.

Q→ Test convergence of series :-

$$\frac{\sqrt{1}}{1+\sqrt{1}} + \frac{\sqrt{2}}{2+\sqrt{2}} + \frac{\sqrt{3}}{3+\sqrt{3}} + \text{-----}$$

Sol $\Sigma u_n = \frac{\sqrt{1}}{1+\sqrt{1}} + \frac{\sqrt{2}}{2+\sqrt{2}} + \frac{\sqrt{3}}{3+\sqrt{3}} + \text{----}$

$$u_n = \frac{\sqrt{n}}{n+\sqrt{n}}$$

Now $v_n = \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$ [n^{th} term of Aux. Series Σv_n]

Now, $\frac{u_n}{v_n} = \frac{\sqrt{n}}{n+\sqrt{n}} \cdot \frac{\sqrt{n}}{1} = \frac{n}{n+\sqrt{n}}$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n}{n+\sqrt{n}} = \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n}\left[1+\frac{1}{\sqrt{n}}\right]} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{\sqrt{n}}}$$

$= 1$

(fixed, finite, Non zero)

⇒ Comparision Test is applicable

By p-test, $\Sigma v_n = \Sigma \frac{1}{\sqrt{n}}$ is divergent as $p = \frac{1}{2} < 1$

⇒ Σu_n is divergent

Q→ Test for convergence whose n^{th} term is given.

$$\Sigma u_n = \Sigma \frac{2n^2+1}{3n^3+5n^2+6}$$

Sol $u_n = \frac{2n^2+1}{3n^3+5n^2+6}$

Now $v_n = \frac{n^2}{n^3} = \frac{1}{n}$ [n^{th} term of Aux Series Σv_n]

Now $\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{2n^2+1}{3n^3+5n^2+6} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{\cancel{n^3}\left[2+\frac{1}{n^2}\right]}{\cancel{n^3}\left[3+\frac{5}{n}+\frac{6}{n^3}\right]}$

$= 2/3$

(fixed, finite, Non zero)

By p-test $\Sigma V_n = \Sigma \frac{1}{n}$ divergent as $p=1$

$\Rightarrow \Sigma u_n$ is divergent.

Q→ Test convergence of series

$$\frac{1}{1\cdot 2\cdot 3} + \frac{3}{2\cdot 3\cdot 4} + \frac{5}{3\cdot 4\cdot 5} + \cdots$$

Sol $\Sigma u_n = \frac{1}{1\cdot 2\cdot 3} + \frac{3}{2\cdot 3\cdot 4} + \frac{5}{3\cdot 4\cdot 5} + \cdots$

$$u_n = \frac{2n-1}{n(n+1)(n+2)}$$

$$v_n = \frac{n}{n^3} = \frac{1}{n^2} \quad \left[n^{th} \text{ term of Aux. series } \Sigma V_n\right]$$

1, 3, 5 - - -
n^{th} term $= 1+(n-1)2$
$2n-1$

1, 2, 3 - - -
n^{th} term $= n$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{2n-1}{n(n+1)(n+2)} \cdot \frac{n^2}{1}$$

$$= \lim_{n\to\infty} \frac{\cancel{n^2}\cdot \cancel{n}\left[2-\frac{1}{n}\right]}{\cancel{n}\cdot \cancel{n}\left(1+\frac{1}{n}\right)\cancel{n}\left(1+\frac{2}{n}\right)}$$

$$= \lim_{n\to\infty} \frac{\left[2-1/n\right]}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)} = \frac{2}{1} = 2 \text{ (fixed, finite Nonzero)}$$

$\Rightarrow$ Comparision Test is applicable.

By p-test, $\Sigma V_n = \Sigma \frac{1}{n^2}$ is convergent as $p=2>1$

$\Rightarrow \Sigma u_n$ is convergent.

Q→ Test the convergence of series : $\sum\left(\sqrt[3]{n^3+1}-n\right)$

Sol $\sum\left(\sqrt[3]{n^3+1}-n\right)$

$$u_n = (n^3+1)^{1/3} - n = \left[n^3\left(1+\frac{1}{n^3}\right)\right]^{1/3} - n$$

$$= n\left[\left(1+\frac{1}{n^3}\right)^{1/3} - 1\right] = n\left[\cancel{1} + \frac{1}{3}\frac{1}{n^3} + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\left(\frac{1}{n^3}\right)^2 + \cdots - \cancel{1}\right]$$

$$= \frac{n}{n^3}\left[\frac{1}{3} + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\frac{1}{n^3} \cdots\right] = \frac{1}{n^2}\left[\frac{1}{3} - \frac{1}{9}\frac{1}{n^3} \cdots\right]$$

Let $v_n = \frac{1}{n^2}$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{\cancel{\frac{1}{n^2}}\left[\frac{1}{3}-\frac{1}{9}\frac{1}{n^3}\cdots\right]}{\cancel{\frac{1}{n^2}}} = \frac{1}{3} \text{ (finite, fixed non zero)}$$

⇒ Comparision Test is applicable

By p-test, $\sum v_n = \sum\frac{1}{n^2}$ is convergent as $p=2$

⇒ $\sum u_n$ is also cgt.

Practice Questions

(1) Test convergence of $\sum\frac{1}{\sqrt{n}+\sqrt{n-1}}$ [Ans:- Divergent]

(2) Test convergence of $\frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \cdots$ [Ans:- divergent]

(3) Test convergence of $\frac{1}{1\cdot 2} + \frac{2}{3\cdot 4} + \frac{3}{5\cdot 6} + \cdots$ [Ans:- Divergent]

D'Alembert Ratio Test

Statement :- If Σu_n is a positive term series such that $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = k$ then

$\Rightarrow$ series is convergent if $k<1$

$\Rightarrow$ series is divergent if $k>1$

$\Rightarrow$ Test fails if $k=1$

OR

$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = l$ then

$\Rightarrow$ convergent if $l>1$

$\Rightarrow$ divergent if $l<1$

$\Rightarrow$ Test fails if $l=1$

Q:→ Prove that $1+\frac{2}{3}+\frac{4}{9}+\frac{8}{27}+\frac{16}{81}$ - - - - converges.

So

$$u_n = \left(\frac{2}{3}\right)^{n-1}$$

$$u_{n+1} = \left(\frac{2}{3}\right)^{n}$$

$$\frac{u_{n+1}}{u_n} = \left(\frac{2}{3}\right)^{n} \cdot \left(\frac{3}{2}\right)^{n-1} \Rightarrow \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{2}{3} = \frac{2}{3} < 1$$

$\Rightarrow \Sigma u_n$ is convergent.

Q → Test the series

$$\frac{2}{1^2+1} + \frac{2^2}{2^2+1} + \frac{2^3}{3^2+1} + \cdots$$

Sol $\quad U_n = \dfrac{2^n}{n^2+1}$

$$U_{n+1} = \frac{2^{n+1}}{(n+1)^2+1}$$

$$\frac{U_{n+1}}{U_n} = \left[\frac{2^{n+1}}{(n+1)^2+1}\right] \cdot \frac{n^2+1}{2^n} = \frac{2\,\cancel{n^2}\left(1+\frac{1}{n^2}\right)}{\cancel{n^2}\left[\left(1+\frac{1}{n}\right)^2+\frac{1}{n^2}\right]}$$

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} \frac{2\left(1+\frac{1}{n^2}\right)}{\left[\left(1+\frac{1}{n}\right)^2+\frac{1}{n^2}\right]} = \frac{2}{1} = 2 > 1$$

$\Rightarrow \sum U_n$ is divergent.

Q → Test the series

$$1 + \frac{x}{2} + \frac{x^2}{5} + \frac{x^3}{10} + \cdots + \frac{x^n}{n^2+1} + \cdots$$

Sol Leaving first term,

$$U_n = \frac{x^n}{n^2+1}$$

$$U_{n+1} = \frac{x^{n+1}}{(n+1)^2+1}$$

$$\frac{U_{n+1}}{U_n} = \frac{x^{n+1}}{\left[(n+1)^2+1\right]} \frac{\left[n^2+1\right]}{x^n}$$

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} x \cdot \frac{\cancel{n^2}\left(1+\frac{1}{n^2}\right)}{\cancel{n^2}\left[\left(1+\frac{1}{n}\right)^2+1\right]} = x$$

If $x < 1$ then convergent
$x > 1$ then divergent

if $x = 1$ Test fails.

when $x=1$, then

$$u_n = \frac{1}{n^2+1}$$

$$v_n = \frac{1}{n^2}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{n^2+1} \cdot \frac{n^2}{1} = \lim_{n\to\infty} \frac{\cancel{n^2}}{\cancel{n^2}\left[1+\frac{1}{n^2}\right]} = 1$$

$\Rightarrow$ comparision Test is applicable

By p-test, $\Sigma v_n = \Sigma \frac{1}{n^2}$ is convergent as $p=2>1$

$\Rightarrow \Sigma u_n$ is convergent.

Finally, series is convergent if $x \le 1$
and " " divergent if $x>1$.

Practice Questions

(1) Test the series:- $\frac{1}{1\cdot 2\cdot 3} + \frac{x}{4\cdot 5\cdot 6} + \frac{x^2}{7\cdot 8\cdot 9} + \cdots$ [Ans:- cgt, $x\le 1$; dgt $x>1$

(2) Test series $\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \cdots$ [Ans:- cgt $x^2 \le 1$; dgt $x^2>1$

Raabe's Test

If Σu_n is positive term series then.

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right)=k,$$

$\Rightarrow$ series is convergent if $k>1$

$\Rightarrow$ " " divergent if $k<1$

$\Rightarrow$ Test fails if $k=1$.

Note: $\rightarrow$ Raabe's Test is applied only when D'Alembert Ratio test fails. (i.e $\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=1$)

Q $\rightarrow$ Test the convergence of series:-

$$1+\frac{2}{3}\cdot\frac{1}{4}+\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{1}{6}+\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdot\frac{1}{8}+\cdots$$

Sol Leaving first term,

$$u_n=\frac{2\cdot4\cdot6\cdots2n}{3\cdot5\cdot7\cdots(2n+1)}\;\frac{1}{2n+2}$$

3, 5, 7 - - -
$n^{th} = 3+(n-1)2$
$2n+1$

4, 6, 8 - - - -
$n^{th} = 4+(n-1)\cdot2$
$2n+2$

$$u_{n+1}=\frac{2\cdot4\cdot6\cdots2n\,(2n+2)}{3\cdot5\cdot7\cdots(2n+1)(2n+3)}\;\frac{1}{2n+4}$$

$$\frac{u_n}{u_{n+1}}=\frac{\cancel{2\cdot4\cdot6\cdots2n}}{\cancel{3\cdot5\cdot7\cdots(2n+1)}}\;\frac{1}{2n+2}\;\cdot\frac{\cancel{3\cdot5\cdot7\cdots(2n+1)}(2n+3)\;(2n+4)}{\cancel{2\cdot4\cdot6\cdots2n}(2n+2}$$

$$=\frac{(2n+3)(2n+4)}{(2n+2)(2n+2)}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\frac{(2n+3)(2n+4)}{(2n+2)(2n+2)}$$

$$=\lim_{n\to\infty}\frac{\cancel{n}\left(2+\frac{3}{n}\right)\cancel{n}\left(2+\frac{4}{n}\right)}{\cancel{n}\left(2+\frac{2}{n}\right)\cancel{n}\left(2+\frac{2}{n}\right)}=\frac{2\cdot2}{2\cdot2}$$

$$=1$$

Test fails.

[Ratio Test]

Ratio Test
$\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=l$
$\Rightarrow l>1 \Rightarrow cgt$
$l<1 \Rightarrow dgt$
$l=1 \Rightarrow$ Test fails.

Now $\frac{u_n}{u_{n+1}} - 1 = \frac{(2n+3)(2n+4)}{(2n+2)(2n+2)} - 1$

$= \frac{(2n+3)(2n+4) - (2n+2)^2}{(2n+2)^2}$

$= \frac{6n+8}{(2n+2)^2}$

Now $n\left(\frac{u_n}{u_{n+1}} - 1\right) = n\left(\frac{6n+8}{(2n+2)^2}\right) = \frac{6n^2+8n}{(2n+2)^2}$

$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} \frac{6n^2+8n}{(2n+2)^2} = \lim_{n\to\infty} \frac{\cancel{n^2}\left(6+\frac{8}{n}\right)}{\cancel{n^2}\left(2+\frac{2}{n}\right)^2}$

$= \frac{6}{4} = \frac{3}{2} > 1$ [Raabe's Test]

$\Rightarrow \Sigma u_n$ is convergent.

Q→ Test convergence of series.

$\frac{x}{1\cdot 2} + \frac{x^2}{3\cdot 4} + \frac{x^3}{5\cdot 6} + \frac{x^4}{7\cdot 8} + \cdots$

Sol $u_n = \frac{x^n}{(2n-1)\,2n}$

1, 3, 5 ---
n^{th} term $= 1+(n-1)2$
$2n-1$

$u_{n+1} = \frac{x^{n+1}}{(2n+1)(2n+2)}$

$\frac{u_n}{u_{n+1}} = \frac{x^n}{(2n-1)\,2n} \cdot \frac{(2n+1)(2n+2)}{x^{n+1}} = \frac{(2n+1)(2n+2)}{(2n-1)\,2n} \cdot \frac{1}{x}$

$\lim_{n\to\infty} \frac{(2n+1)(2n+2)}{(2n-1)\,2n} \frac{1}{x} = \lim_{n\to\infty} \frac{\cancel{n}\left(2+\frac{1}{n}\right)\cancel{n}\left(2+\frac{2}{n}\right)}{\cancel{n}\left(2-\frac{1}{n}\right)2\cancel{n}} \frac{1}{x}$

$= \frac{2\cdot 2}{2\cdot 2}\frac{1}{x} = \frac{1}{x}$

Now series is cgt if $\frac{1}{x} > 1 \Rightarrow x < 1$

" " dgt if $\frac{1}{x} < 1 \Rightarrow x > 1$

Test fails if $\frac{1}{x} = 1 \Rightarrow x = 1$

When $x = 1$

$$\frac{u_n}{u_{n+1}} = \frac{(2n+1)(2n+2)}{(2n-1)\,2n}$$

$$n\left(\frac{U_n}{U_{n-1}} - 1\right) = n\left[\frac{(2n+1)(2n+2)}{(2n-1)\,2n} - 1\right] = n\left[\frac{(2n+1)(2n+2) - 2n(2n-1)}{(2n)(2n-1)}\right]$$

$$= n\left[\frac{8n+2}{2n(2n-1)}\right] = \frac{8n^2+2n}{2n(2n-1)}$$

Raabe's Test

$$\lim_{n\to\infty} n\left[\frac{U_n}{U_{n+1}} - 1\right] = \lim_{n\to\infty} \frac{8n^2+2n}{2n(2n-1)}$$

$$= \lim_{n\to\infty} \frac{n^2\left[8 + \frac{2}{n}\right]}{2n\cdot n\left(2 - \frac{1}{n}\right)} = \frac{8}{4} = 2 > 1$$

$\Rightarrow$ series is cgt at $x = 1$

Finally, series is cgt if $x \le 1$
dgt. if $x > 1$

Practice Questions

(1) Test series convergence :- $\frac{x}{1} + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} + \cdots$

[Ans :- cgt, $x^2 \le 1$
dgt, $x^2 > 1$

(2) Test convergence

$1 + \frac{x}{2} + \frac{1\cdot 3}{2\cdot 4}x^2 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3 + \cdots$

[Ans :- cgt $x < 1$
dgt $x \geq 1$

FOURIER SERIES

FOURIER SERIES FOR THE FUNCTION $f(x)$ in the interval $c < x < c+2\pi$ is given by

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx}$$

where,

$$a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx$$

Above formulae are also called Euler's Formulae. Constants a_0, a_n and b_n are called Fourier Cofficients of $f(x)$

Cor. 1 → If $c=0$, the interval becomes $0 < x < 2\pi$ and the formulae reduce to

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

Cor. 2 → If $c = -\pi$, the interval becomes $-\pi < x < \pi$ and the formulae reduce to

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

DIRICHLET'S CONDITIONS

Any function $f(x)$ can be expressed as a Fourier series $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$, where a_0, a_n, b_n are constants, provided

i) $f(x)$ is periodic, single valued function
ii) $f(x)$ has a finite number of finite discontinuities in any one period.
iii) $f(x)$ has a finite number of Maxima and Minima.

When these conditions are satisfied, the Fourier series converges to $f(x)$ at every point of continuity. At a point of discontinuity the sum of series is equal to mean of limits on right and left i.e. $\frac{1}{2}[f(x+0) + f(x-0)]$

Question If $f(x) = \left(\frac{\pi - x}{2}\right)^2, \quad 0 < x < 2\pi,$ then show that

$$f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx \quad [2017]$$

Solution By Euler's formulae, we have

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} \left(\frac{\pi - x}{2}\right)^2 dx$$

$$a_0 = \frac{1}{4\pi} \left[\frac{(\pi - x)^3}{3} (-1) \right]_0^{2\pi}$$

$$a_0 = -\frac{1}{12\pi} \left[-\pi^3 - \pi^3 \right] = \boxed{\frac{\pi^2}{6}}$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} \left(\frac{\pi - x}{2}\right)^2 \cos nx \, dx$$

$$= \frac{1}{4\pi} \left[(\pi - x)^2 \frac{\sin nx}{n} - 2(\pi - x)(-1)\left(-\frac{\cos nx}{n^2}\right) + 2(1)\left(-\frac{\sin nx}{n^3}\right) \right]_0^{2\pi}$$

$$= \frac{1}{4\pi} \left[-\frac{2}{n^2} \left\{ -\pi \cos 2n\pi - \pi \cos 0 \right\} \right]$$

$$[\because \sin 2n\pi = 0, \ \sin 0 = 0]$$

$$= \frac{1}{4\pi} \left[-\frac{2}{n^2} \left\{ -2\pi \right\} \right]$$

$$\boxed{a_n = \frac{1}{n^2}} \qquad [\because \cos 2n\pi = 1, \ \cos 0 = 1]$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi-x}{2}\right)^2 \sin nx \, dx$$

$$= \frac{1}{4\pi}\left[(\pi-x)^2\left\{\frac{-\cos nx}{n}\right\} - 2(\pi-x)(-1)\left\{\frac{-\sin nx}{n^2}\right\} + 2(-1)(-1)\left\{-\frac{\cos nx}{n^3}\right\}\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\frac{-\pi^2}{n}\cos 2n\pi + \frac{\pi^2}{n}\cos 0 - 0 - \frac{2}{n^3}\left\{\cos 2n\pi - \cos 0\right\}\right]$$

$$[\because \sin 0 = \sin 2n\pi = 0]$$

$$= \frac{1}{4\pi}[0] \quad [\because \cos 0 = \cos 2n\pi = 1]$$

$$\boxed{b_n = 0}$$

Now,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$\boxed{\therefore \quad f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{\cos nx}{n^2}}$$ Hence Proved

PRACTICE QUESTION

Question → Obtain Fourier Series for $f(x) = \left(\frac{\pi - x}{2}\right)$ in interval $(0, 2\pi)$ [2009]

and hence deduce

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

EVEN AND ODD FUNCTIONS

A function is said to be Even if $f(-x) = f(x)$

Ex→ x^2, $\cos x$, $\sin^2 x$ are Even functions

A function is said to be odd if $f(-x) = -f(x)$

Ex→ x^3, $\sin x$, $\tan^3 x$ are odd functions

#

$$\int_{-\pi}^{\pi} f(x)\,dx = \begin{cases} 2\int_0^{\pi} f(x)\,dx, & \text{If } f(x) \text{ is Even func}^n \\ 0, & \text{If } f(x) \text{ is odd func}^n \end{cases}$$

Even

odd

CHAIN RULE

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots$$

where u and v are functions of x and dashes denote differentiation and suffixes denotes integration w.r.to x.

#

$\sin 0 = 0$,	$\cos 0 = 1$
$\sin n\pi = 0$,	$\cos n\pi = (-1)^n$
$\sin 2n\pi = 0$,	$\cos 2n\pi = 1$

Question → Expand the Function $f(x) = x \sin x$ as a Fourier Series in the interval $-\pi \le x \le \pi$. Hence deduce that $\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \frac{1}{7\cdot 9} + \cdots = \frac{\pi - 2}{4}$

[2015]

Solution $f(x) = x \sin x$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin x \, dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x \sin x \, dx$$

[$\because$ $x \sin x$ is an even function]

$$= \frac{2}{\pi}\left[x(-\cos x) - 1(-\sin x)\right]_0^{\pi}$$

$$= \frac{2}{\pi}(\pi)$$

$$\boxed{a_0 = 2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin x \cos nx \, dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x \sin x \cos nx \, dx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} x \left\{ \sin(n+1)x - \sin(n-1)x \right\} dx$$

[$\because$ $2\cos A \sin B = \sin(A+B) - \sin(A-B)$]

$$= \frac{1}{\pi}\int_{0}^{\pi} x \sin(n+1)x \, dx - \frac{1}{\pi}\int_{0}^{\pi} x \sin(n-1)x \, dx$$

$$= \frac{1}{\pi}\left[x\left\{-\frac{\cos(n+1)x}{n+1}\right\}_0^{\pi} - (1)\left\{-\frac{\sin(n+1)x}{(n+1)^2}\right\}\right]_0^{\pi}$$

$$- \frac{1}{\pi}\left[x\left\{-\frac{\cos(n-1)x}{n-1}\right\} - (1)\left\{-\frac{\sin(n+1)x}{(n-1)^2}\right\}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\pi\frac{(-1)^{n+1}}{n+1} + 0\right] - \frac{1}{\pi}\left[-\pi\frac{(-1)^{n-1}}{n-1} - 0\right]$$

$$= \frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1}$$

$$= (-1)^{n+1}\left[\frac{-1}{n+1} + \frac{1}{n-1}\right] = \boxed{\frac{2(-1)^{n+1}}{n^2-1}}$$

$[n \neq 1]$

$$a_1 = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos x \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x \sin 2x \, dx$$

$$= \frac{1}{\pi}\left[x\left\{-\frac{\cos 2x}{2}\right\} - (1)\left\{-\frac{\sin 2x}{4}\right\}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{2}\right]$$

$$\boxed{a_1 = -\frac{1}{2}}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin x \sin nx \, dx$$

$$\boxed{b_n = 0}$$

[$\because$ $x \sin x \sin nx$ is an odd function]

Hence,

$$f(x) = 1 - \frac{1}{2}\cos x + 2\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n^2-1}\cos nx$$

$$x\sin x = 1 - \frac{1}{2}\cos x + 2\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{(n-1)(n+1)}\cos nx$$

$$x\sin x = 1 + 2\left[-\frac{1}{4}\cos x - \frac{1}{1\cdot 3}\cos 2x + \frac{1}{2\cdot 4}\cos 3x - \frac{1}{3\cdot 5}\cos 4x + \cdots\right]$$

Putting $x = \pi/2$ we get,

$$\frac{\pi}{4} = \frac{1}{2} + \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots$$

$$\Rightarrow \frac{\pi}{4} - \frac{1}{2} = \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots$$

$$\Rightarrow \frac{\pi-2}{4} = \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots$$

PRACTICE QUESTION

Question → Find Fourier Series for the function $f(x) = x\cos x$, $-\pi < x < \pi$

Answer → $x\cos x = -\frac{1}{2}\sin x + \frac{4\sin 2x}{2^2-1} - \frac{6\sin 3x}{3^2-1} + \cdots$

[Hint → $a_0 = a_n = 0$, $b_n = \left\{-\frac{(-1)^{n+1}}{n+1} - \frac{(-1)^{n+1}}{n-1}\right\}$ $n \neq 1$

$b_1 = -\frac{1}{2}$]

Question → Find the Fourier Series Expansion of the periodic function of period 2π

$$f(x) = x^2, \quad -\pi \le x \le \pi$$

Hence, find the sum of the series [2004]

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \text{------}$$

Solution $\because f(x) = x^2, \; -\pi \le x \le \pi$

This is an Even function

$\therefore \boxed{b_n = 0}$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\,dx$$

[$\because x^2$ is an Even func^n]

$$= \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$$

$$\boxed{a_0 = \frac{2\pi^2}{3}}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,dx$$

$$= \frac{2}{\pi}\left[x^2\left\{\frac{\sin nx}{n}\right\} - (2x)\left\{-\frac{\cos nx}{n^2}\right\} + 2\left\{-\frac{\sin nx}{n^3}\right\}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi^2 \sin n\pi}{n} + \frac{2\pi \cos n\pi}{n^2} - \frac{2\sin n\pi}{n^3}\right]$$

$$a_n = \frac{2}{\pi}\left[\frac{2\pi}{n^2}(-1)^n\right]$$

$$\boxed{a_n = \frac{4(-1)^n}{n^2}}$$

FOURIER SERIES IS

$$f(x) = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + a_3\cos 3x + \cdots$$

$$x^2 = \frac{\pi^2}{3} - 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \cdots\right]$$

Put $x = 0$, we get

$$0 = \frac{\pi^2}{3} - 4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\right]$$

$$\Rightarrow \boxed{\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}}$$ Answer

PRACTICE QUESTIONS

Question → Obtain a Fourier Expression for $f(x) = x^3$, $-\pi < x < \pi$

Answer → $f(x) = 2\sum_{n=1}^{\infty}(-1)^n\left[-\frac{\pi^2}{n} + \frac{6}{n^3}\right]\sin nx$

Question → Find a Fourier Series for Function $f(x) = x - x^2$, $-\pi \le x \le \pi$

Hence show that $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$

[Hint → $a_0 = -\frac{2}{3}\pi^2$, $a_n = -\frac{4}{n^2}(-1)^n$, $b_n = -\frac{2}{n}(-1)^n$]

Question: Obtain Fourier Series for the function $f(x) = \begin{cases} x ; & -\pi < x < 0 \\ -x ; & 0 < x < \pi \end{cases}$

and hence show that [2002, 2013, 2015, 2018]

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

Solution: $\because$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad (1)$$

Where,

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{0} x\, dx + \int_{0}^{\pi} -x\, dx \right]$$

$$= \frac{1}{\pi} \left[\left\{ \frac{x^2}{2} \right\}_{-\pi}^{0} - \left\{ \frac{x^2}{2} \right\}_{0}^{\pi} \right]$$

$$\boxed{a_0 = -\pi}$$

$$a_n = \frac{1}{\pi} \left[\int_{-\pi}^{0} x \cos nx\, dx + \int_{0}^{\pi} -x \cos nx\, dx \right]$$

$$= \frac{1}{\pi} \left[x \left\{ \frac{\sin nx}{n} \right\} - 1 \cdot \left\{ -\frac{\cos nx}{n^2} \right\} \right]_{-\pi}^{0}$$

$$- \frac{1}{\pi} \left[x \left\{ \frac{\sin nx}{n} \right\} - 1 \cdot \left\{ -\frac{\cos nx}{n^2} \right\} \right]_{0}^{\pi}$$

$$a_n = \frac{1}{\pi n^2}\left[(\cos 0 - \cos n\pi) - (\cos n\pi - \cos 0)\right]$$

$$a_n = \frac{1}{\pi n^2}\left[2 - 2\cos n\pi\right]$$

$$a_n = \frac{2}{\pi n^2}\left[1 - (-1)^n\right] = \begin{cases} 0, & n \text{ is even} \\ \frac{4}{\pi n^2}, & n \text{ is odd} \end{cases}$$

Even / odd

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} x \sin nx\, dx + \int_{0}^{\pi} (-x)\sin nx\, dx\right]$$

$$b_n = \frac{1}{\pi}\left[x\left\{-\frac{\cos nx}{n}\right\} - 1\cdot\left\{-\frac{\sin nx}{n^2}\right\}\right]_{-\pi}^{0}$$

$$- \frac{1}{\pi}\left[x\left\{-\frac{\cos nx}{n}\right\} - 1\cdot\left\{-\frac{\sin nx}{n^2}\right\}\right]_{0}^{\pi}$$

$$= -\frac{1}{n\pi}\left[\pi\cos n\pi\right] + \frac{1}{n\pi}\left[\pi\cos n\pi\right] = 0$$

From (1)

$$f(x) = -\frac{\pi}{2} + \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right] \quad (2)$$

At point of discontinuity

$$f(0) = \frac{1}{2}\left[f(0-0) + f(0+0)\right]$$

$$= \frac{1}{2}(0-0) = 0$$

Putting $x=0$ in Equation (2), we get

$$0 = -\frac{\pi}{2} + \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$$

$$\Rightarrow \boxed{\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}}$$

Hence Proved

PRACTICE QUESTIONS

Question → Find Fourier series to represent the function

$$f(x) = \begin{cases} -K, & -\pi < x < 0 \\ K, & 0 < x < \pi \end{cases}$$

Also deduce that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

[2010, 2016]

Answer → $f(x) = \frac{4K}{\pi}\left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right]$

Question → Find Fourier series for the function

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$$

Hence find the sum of series

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

[2013, 2014, 2016]

Answer → a) series → $f(x) = \frac{\pi}{4} - \frac{2}{\pi}\sum_{\substack{n=1 \\ (n \text{ is odd})}}^{\infty} \frac{1}{n^2}\cos nx - \sum_{n=1}^{\infty} \frac{(-1)^n}{n}\sin nx$

b) sum → $\frac{\pi^2}{8}$

Question → Expand $f(x) = x\sin x$ as a fourier series in $0 < x < 2\pi$ [2015]

Answer → $f(x) = -1 - \frac{1}{2}\cos x + \pi \sin x + \sum_{n=2}^{\infty} \frac{2}{n^2-1}\cos nx$

Change of Interval

Fourier Series $f(x)$ in the interval $c < x < c+2l$ is given by

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}}$$

where,

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx,$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\,dx,$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

Cor. 1 → If we put $c=0$, the interval becomes $0 < x < 2l$, then

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx,$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx,$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

Cor. 2 → If we put $c=-l$, the interval becomes $-l < x < l$, then

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

Question → Obtain the Fourier Series Expansion of $f(x) = \left(\frac{\pi - x}{2}\right)$ for $0 < x < 2$ [2011]

Solution → $\because f(x) = \left(\frac{\pi - x}{2}\right)$, $l = 1$

$$\therefore a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx = \int_0^2 \left(\frac{\pi - x}{2}\right) dx$$

$$= \frac{1}{2}\left[\pi x - \frac{x^2}{2}\right]_0^2 = \frac{1}{2}(2\pi - 2)$$

$$\boxed{a_0 = \pi - 1}$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \int_0^2 \left(\frac{\pi - x}{2}\right)\cos n\pi x\,dx$$

$$= \left[\left(\frac{\pi - x}{2}\right)\frac{\sin n\pi x}{n\pi} - \left(-\frac{1}{2}\right)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right]_0^2$$

$$= -\frac{1}{2n^2\pi^2}\left[\cos n\pi x\right]_0^2 = \boxed{0}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx = \int_0^2 \left(\frac{\pi - x}{2}\right)\sin n\pi x\,dx$$

$$= \left[\left(\frac{\pi - x}{2}\right)\left(-\frac{\cos n\pi x}{n\pi}\right) - \left(-\frac{1}{2}\right)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_0^2$$

$$= -\frac{1}{2n\pi}\left[(\pi - 2) - \pi\right]$$

$$\boxed{b_n = \frac{1}{n\pi}}$$

Now, Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$\Rightarrow \boxed{\frac{\pi - x}{2} = \frac{\pi - 1}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin n\pi x} \quad \text{Answer}$$

PRACTICE QUESTION

Question: Find Fourier series for the function $f(x) = \begin{cases} x, & 0 < x < 1 \\ 1 - x, & 1 < x < 2 \end{cases}$

[2011]

Answer

$$f(x) = \frac{-4}{\pi^2}\left[\frac{\cos\pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \cdots\right] + \frac{2}{\pi}\left[\frac{\sin\pi x}{1} + \frac{\sin 3\pi x}{3} + \cdots\right]$$

Half Range Series

A Function $f(x)$ defined over the interval $0 < x < l$ is capable of two distinct half range series.

The half range cosine series is

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}}$$

Where, $a_0 = \frac{2}{l}\int_0^l f(x)\,dx$;

$$a_n = \frac{2}{l}\int_0^l f(x) \cos \frac{n\pi x}{l}\,dx$$

The half range Sine Series is

$$\boxed{f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}}$$

Where, $b_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l}\,dx$

Cor. 1 → If the range is $0 < x < \pi$, then

i) The half range cosine series is

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx}$$

Where, $a_0 = \frac{2}{\pi}\int_0^\pi f(x)\,dx$;

$$a_n = \frac{2}{\pi}\int_0^\pi f(x) \cos nx\,dx$$

ii) The half-range Sine Series is

$$\boxed{f(x) = \sum_{n=1}^{\infty} b_n \sin nx}$$

where, $b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$

Question → Find half range Sine series for the function $f(x) = x^2$ in interval $0 < x < 3$

[2015]

Solution → ∵ Half range Sine Series is given by $f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$

where $b_n = \frac{2}{l}\int_0^{l} f(x) \sin \frac{n\pi x}{l} dx$

Here, we have half range $0 < x < 3$

and $f(x) = x^2$

$$\therefore b_n = \frac{2}{3}\int_0^3 x^2 \sin \frac{n\pi x}{3} dx$$

$$= \frac{2}{3}\left[x^2\left(\frac{3}{n\pi}\right)\left(-\cos\frac{n\pi x}{3}\right) + 2x\left(\frac{9}{n^2\pi^2}\right)\left(\sin\frac{n\pi x}{3}\right) - 2\left(\frac{27}{n^3\pi^3}\right)\left(-\cos\frac{n\pi x}{3}\right)\right]_0^3$$

$$= \frac{2}{3}\left[\left\{-\frac{27}{n\pi}(-1)^n - \frac{54}{n^3\pi^3}(-1)^n\right\} + \frac{54}{n^3\pi^3}\right]$$

$$= \frac{2}{3}\left[\frac{54}{n^3\pi^3}\left\{1-(-1)^n\right\} - \frac{27}{n\pi}(-1)^n\right]$$

$$\Rightarrow \quad b_n = \frac{2}{3}\left[\frac{108}{n^3\pi^3} + \frac{27}{n\pi}\right], \text{ when } n \text{ is odd}$$

and $$b_n = \frac{-18}{n\pi}, \quad \text{when } n \text{ is Even}$$

$\therefore$ Half range Sine series is

$$f(x) = b_1 \sin\frac{\pi x}{3} + b_2 \sin\frac{2\pi x}{3} + b_3 \sin\frac{3\pi x}{3} + \cdots$$

$$= \frac{2}{3}\left[\frac{108}{\pi^3}\left(\frac{\sin\frac{\pi x}{3}}{1^3} + \frac{\sin\frac{3\pi x}{3}}{3^3} + \cdots\right)\right.$$

$$+ \frac{27}{\pi}\left[\frac{\sin\frac{\pi x}{3}}{1} + \frac{\sin\frac{3\pi x}{3}}{3} + \cdots\right]$$

$$\left. -\frac{18}{\pi}\left(\frac{\sin\frac{2\pi x}{3}}{2} + \frac{\sin\frac{4\pi x}{3}}{4} + \cdots\right)\right]$$

Question $\rightarrow$ Find fourier half range cosine series of the function

$$f(t) = \begin{cases} 2t, & 0 < t < 1 \\ 2(2-t), & 1 < t < 2 \end{cases}$$

[2002, 2006, 2007, 2008, 2017]

Solution $\rightarrow$ Half range cosine Series is

$$f(t) = \frac{a_0}{2} + a_1 \cos\frac{\pi t}{l} + a_2 \cos\frac{2\pi t}{l} + a_3 \cos\frac{3\pi t}{l} + \cdots + b_1 \sin\frac{\pi t}{l} + b_2 \sin\frac{2\pi t}{l} + \cdots \quad \text{--- (1)}$$

Here $l = 2$

Hence $a_0 = \frac{2}{l}\int_0^l f(t)\,dt$

$$= \frac{2}{2}\int_0^1 2t\,dt + \frac{2}{2}\int_1^2 2(2-t)\,dt$$

$$= \left[t^2\right]_0^1 + 2\left[2t - \frac{t^2}{2}\right]_1^2$$

$$\boxed{a_0 = 2}$$

$$a_n = \frac{2}{l}\int_0^l f(t)\cos\frac{n\pi t}{l}\,dt$$

$$= \frac{2}{2}\int_0^1 2t\cos\frac{n\pi t}{2}\,dt + \frac{2}{2}\int_1^2 2(2-t)\cos\frac{n\pi t}{2}\,dt$$

$$= \left[2t\left\{\frac{2}{n\pi}\sin\frac{n\pi t}{2}\right\} - (2)\left\{\frac{-4}{n^2\pi^2}\cos\frac{n\pi t}{2}\right\}\right]_0^1$$

$$+ \left[(4-2t)\left\{\frac{2}{n\pi}\sin\frac{n\pi t}{2}\right\} - (-2)\left\{\frac{-4}{n^2\pi^2}\cos\frac{n\pi t}{2}\right\}\right]_1^2$$

$$= \left[\frac{4}{n\pi}\sin\frac{n\pi}{2} + \frac{8}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{8}{n^2\pi^2}\right]$$

$$+ \left[0 - \frac{8}{n^2\pi^2}\cos n\pi - \frac{4}{n\pi}\sin\frac{n\pi}{2} + \frac{8}{n^2\pi^2}\cos\frac{n\pi}{2}\right]$$

$$= \frac{16}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{8}{n^2\pi^2} - \frac{8}{n^2\pi^2}\cos\pi$$

$$= \frac{8}{n^2\pi^2}\left[2\cos\frac{n\pi}{2} - 1 - \cos n\pi\right]$$

$$\therefore \boxed{f(t) = 1 + \frac{8}{\pi^2}\sum_{n=1}^{\infty}\left[2\cos\frac{n\pi}{2} - 1 - \cos n\pi\right]\cos\frac{n\pi t}{2}}$$

Answer

PRACTICE QUESTIONS

Question → Find half range sine series of

$$f(x) = \begin{cases} x, & 0 < x < 2 \\ 4-x, & 2 < x < 4 \end{cases}$$

[2015, 2018]

Answer → $b_n = \frac{16}{n^2\pi^2}\sin\frac{n\pi}{2}$,

$$f(x) = \frac{16}{\pi^2}\left[\sin\frac{\pi x}{4} - \frac{1}{3^2}\sin\frac{3\pi x}{4} + \frac{1}{5^2}\sin\frac{5\pi x}{4} - \cdots\right]$$

Question → Find Fourier series of $f(x) = x\sin x$ as cosine series in $(0, \pi)$ and hence show that [2002, 2013, 2017]

$$\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots = \frac{\pi - 2}{4}$$

Question → Find half range sine series of $f(x) = (lx - x^2)$ in Interval $(0, l)$

Hence deduce that $\frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots$

[2012]

Answer → $f(x) = \frac{8l^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin\frac{n\pi x}{l}$, ($n$ is odd)